FLOW OF A DENSE PARTICULATE MIXTURE USING A MODIFIED FORM OF THE MIXTURE THEORY

A. BRIGGS AND C. C. HWANG
Department of Mechanical Engineering
University of Pittsburgh
Pittsburgh, PA 15261

M. MASSOUDI
Federal Energy Technology Center
U.S. Department of Energy
P.O. Box 10940
Pittsburgh, PA 15236

ABSTRACT

The objective of this work is to study the effect of the relative densities of the components in fluid-solid mixture using the formulation presented in Johnson et al., (1991 a,b). The mixture theory is modified to include the effect of buoyancy forces; for this purpose we introduce a term, A_6 v gradρ, in the mechanical interaction between mixture components. When coefficient A_6 is zero, we recover the results presented in Johnson et al., (1991 a,b). It is expected to see settling of the solid particles toward the lower plate which is normally related to the "buoyancy effect" in solid-fluid mixtures. Equations for two-component flows are used to analyze Poiseuille flow between two parallel plates. A review of the basic principles of the mixture theory is presented. The equations for the stress tensor of each component of the mixture and the interactions between the components are given. Flow of a fluid-solid mixture between plates is presented with numerical methods and results.

1. INTRODUCTION

Since there is a wide variety of applications for flow of mixtures in industry, multicomponent systems have become the subject of extensive studies in the last few decades. In this paper, fluid-solid mixtures will refer to a fluid with entrained solid particles, where the fluid can be either a liquid or a gas. Fluidized beds, pneumatic and hydraulic transport of solid particles are some of the examples of the fluid-solid mixtures in industry. Especially the flow behavior of fluid-particle mixtures in transport lines has been of interest in many chemical processes for many years. For example, a great deal of research has been devoted to the use of coal-based slurries as retrofit fuels. In general, empirical relations that predict, for example, the flow and pressure drop in such processes have been developed for specific ranges of solids and fluid properties as well as for various geometries.

Another area of research which has received attention is that of fluidization. Because the fluidized particles are separated from one another by the fluid, intimate contact between the particles and the fluid is achieved. Hence, a bed of fluidized particles is an excellent medium for the transfer of heat or mass between solid particles and fluids. Flowing granular materials has also become an interesting area for studies, since they represent a limiting case of two-phase flow at high solids concentration and high solid-to-fluid density ratios. Many situations, such as discharge through bin outlets, flow through hoppers and chutes, natural phenomena such as avalanches or debris flows present challenging cases. To fully describe and predict the flow and behavior of these complex flows, different multiphase theories have been proposed and used. As indicated by the extensive literature, either averaging or mixture theory is used to model multicomponent systems.

The mixture theory was first presented within the framework of modern continuum mechanics by Truesdell (1984, 1992) who studied the interaction between several constituents by generalizing the equations and principles of a single continuum. The fundamental assumption in this theory is that at any instant of time, every point in space is occupied by one particle from each constituent. Both theories, averaging and mixture require constitutive relations for the stress tensors and for interaction between components, for the case of a purely mechanical problem where thermal, chemical, or electro-magnetic effects are not considered. The historical development of mixture theory can be found in the review articles by Atkin and Craine (1976), Bedford and Drumheller (1983), Bowen (1976), Truesdell (1984), Homsy et al (1980), Ahmadi (1980, 1982), Passman and Nunziato (1986), and Massoudi (1986) where such an approach for modeling fluid-solid systems has been used.

The mixture of fluid and solid particles considered in this paper is assumed to be a purely mechanical system; thermal effects and chemical reactions are ignored. In this theory, which is sometimes also called the theory of interacting continua, it is assumed a priori that each of the components that makeup the mixture are present at each point in space. In other words, each point in space is occupied by a particle belonging to each constituent homogenized over the current configuration. The governing equations are written for each constituent and the interactions between the constituents, e.g., the conversion of one constituent into the other or the supply of mass, momentum, and energy from one constituent to the other, are incorporated into the equations through an appropriate constitutive theory. Mixture theory has been successfully implemented, through the works of Rajagopal, et al., in studying flows of fluids through elastic rubber and flow through porous media. For an excellent and thorough review of these issues, the reader is referred to the book by Rajagopal and Tao (1995).

The two major concerns in the constitutive modeling of two phase flows are the modeling of the stress tensor for the solid phase, and the modeling of the interactive forces. For the latest and on up-to-date review of the existing models for granular materials, i.e., the solid phase, we refer the reader to the work of Hutter and Rajagopal (1994). For a review of the interaction mechanisms we refer the reader to Johnson et al. (1990).

While mixture theory has been rigorously outlined, (cf. Truesdell's Rational Thermodynamics) at times many have found it necessary to modify the theory for a certain application or a class of problems. One such modification which has a relevant and similar relationship to the present work is the study of Katsube and Carroll (1987 a,b). In their analysis of the flow through porous materials, they introduced a new kinematical quantity named porosity, and through this term, the individual stresses were accordingly modified. Similarly, the objective of the present study is to modify or rather extend the mixture theory proposed by Johnson, Massoudi, and Rajagopal (1991 a,b) so that the effect of fluid pressure is included in the equation of motion for the solid particles. This term, is sometimes referred to as the buoyancy force. The present study, in that sense, relies heavily on the works of Johnson, Massoudi, and Rajagopal, and builds on their development and use of mixture theory. A detailed presentation is given in Briggs (1995).

2. THE BASIC EQUATIONS IN MIXTURE THEORY

In this section we provide a summary of the equations of motion for a mixture of a fluid and solid particles. The details are to be found in Rajagopal, et al., (1990), Johnson, et al., (1991 a,b), and Rajagopal and Tao (1994).

The mixture density, ρ_m is given by:

$$\rho_m = \rho_1 + \rho_2, \tag{1}$$

where ρ_1 and ρ_2 are the densities of the mixture components in the current configuration given by:

$$\rho_1 = \Phi \rho_f \qquad \rho_2 = \nu \rho_s, \tag{2}$$

where ρ_f is the density of the pure fluid, ρ_s is the density of the solid particles, and ν is the volume fraction of the solid component and Φ is the volume fraction of the fluid. For a saturated mixture, $\Phi = 1-\nu$. The mean velocity $\mathbf{v}$ of the mixture is:

$$\rho_m \mathbf{v} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2 \tag{3}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the fluid and solid components respectively. Conservation of mass equations for the fluid and solid are:

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 v_1) = 0, \tag{4}$$

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_2 v_2) = 0, \tag{5}$$

It is assumed that there is no mass conversion between two components. If $\mathbf{T}_1$ and $\mathbf{T}_2$ denote partial stress tensors of the fluid and solid, respectively, then linear momentum equations become:

$$\rho_1 \frac{D_1 v_1}{Dt} = \operatorname{div} T_1 + \rho_1 b_1 + f_1 \tag{6}$$

$$\rho_2 \frac{D_2 v_2}{Dt} = \operatorname{div} T_2 + \rho_2 b_2 - \tag{7}$$

where $\mathbf{b}$ represents the body force, and $\mathbf{f}_1$ represents the mechanical interaction between the components. The balance of moment of momentum implies that:

$$T_1 + T_2 = T_1^T + T_2^T \tag{8}$$

The partial stresses need not be symmetric.

In the majority of fluid-solid mixtures, the fluid is either a gas or water. It is appropriate to assume that the fluid behaves as a linearly viscous fluid, with the constitutive equation [Massoudi (1986), Johnson (1991)]:

$$T_f = [-p(\rho_1) + \lambda_f(\rho_1)\operatorname{tr} D_1] I + 2\mu_f(\rho_1) D_1, \tag{9}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities, $\mathbf{D}_1$ is the symmetric part of the velocity gradient tensor for the fluid, and $\mathbf{I}$ is the identity tensor. If the fluid is compressible, then an equation of state is needed for p.

Here, it is assumed that the stress tensor for a granular material is given by [Rajagopal and Massoudi (1990), Goodman and Cowin (1972), Savage (1979)]

$$\begin{aligned} T_s = [\hat{\beta}_0(\rho_2) + \hat{\beta}_1(\rho_2)\operatorname{grad}\rho_2 \bullet \operatorname{grad}\rho_2 + \hat{\beta}_2(\rho_2)\operatorname{tr} D_2] I \\ + \hat{\beta}_3(\rho_2) D_2 + \hat{\beta}_4(\rho_2)\operatorname{grad}\rho_2 \otimes \operatorname{grad}\rho_2 \end{aligned} \tag{10}$$

where $\cdot$ denotes the scalar product of two vectors, and $\otimes$ denotes the outer, or the tensor product of two vectors. Rajagopal and Massoudi (1990, 1994) have outlined an experimental/theoretical approach to determine these material moduli. In the above equation, β_0 is similar to pressure in a compressible fluid, β_1 and β_4 are the material parameters that reflect the distribution of the granular materials, β_2 is akin to the second coefficient of viscosity in a compressible fluid, and β_3 is the viscosity of the granular materials. This model allows for normal-stress differences, a feature which is observed in densely packed granular materials as well as other non-linear fluids or solids. Boyle and Massoudi (1990), using Enskog's dense gas theory, have obtained explicit expressions for the material moduli β_0 through β_4.

We can also define a mixture stress tensor in the following way:

$$T_m = T_1 + T_2, \tag{11}$$

where

$$T_1 = (1-\nu)T_f \ , \quad T_2 = \mathbf{T}_s \tag{12}$$

In this way, the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular material as $\nu \to \nu_m$ where ν_{max} is the solid fraction for maximum packing. $\mathbf{T}_2$ may also be written as $\mathbf{T}_2 = \nu T_s$, where $\mathbf{T}_s$ is the stress tensor for some densely packed reference configuration of the granular materials.

2.1 Interactions

Tchen (1947) synthesized the work of Basset, Boussinesq, Stokes, and Oseen on the motion of a sphere settling under the force of gravity in a fluid at rest. The resulting force balance, sometimes known as the Basset-Boussinesq-Oseen (BBO) equation has been the subject of extensive studies since then (cf. Johnson et. al., (1990) for the references). The basic equation is

$$\begin{aligned} \frac{4\pi a^3}{3}\rho_s \dot{u} &= -\frac{2\pi a^3}{3}\rho_f \dot{u} - 6\pi\mu_f a u \\ &-6\pi\mu_f a \frac{a}{\sqrt{\pi \nu_f}} \int_{-\infty}^{t} \frac{\dot{u}(t_1)}{\sqrt{t-t_1}} dt_1 - \frac{4\pi a^3}{3} g(\rho_s - \rho_f), \end{aligned} \tag{13}$$

where u is the velocity of the particle, $\dot{u}$ is the time derivative of u, ρ_f and ρ_s are the density of the fluid and particle, respectively, a is the particle radius, g is the acceleration of gravity, μ_f and ν_f are viscosity and kinematic viscosity of the fluid respectively. The terms on the right hand side of Equation (13) reflect the presence of virtual mass, Stokes drag, Basset history effects, and buoyancy, respectively. There have been many modifications to this equation to include unsteady flows, flows where the fluid is also in motion, etc. A basic effort in multiphase flow studies has also been to obtain a 'similar' equation where the 'interactive' forces between the two-components (in two-phase flows for example) can be studied. And of course, many studies have started with a modification of this equation to multiphase flows.

Johnson et al. (1990) proposed the following form for the mechanical interaction between the mixture components, $\mathbf{f}_I$:

$$f_1 = A_1 \operatorname{grad} \nu + A_2 F(\nu)(v_2 - v_1) + A_3 \nu (2 \operatorname{tr} D_1^2)^{-1/4} D_1 (v_2 - v_1) + A_4 \nu (W_2 - W_1)(v_2 - v_1) + A_5 a_{vm}, \tag{14}$$

where $\mathbf{a}_{vm}$ is a properly frame invariant measure of the relative acceleration between the mixture components and $F(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The first term reflects the presence of density gradients. The second term in the equation above represents the effect of drag, the third one represents the slip-shear lift or Saffman's lift force and the fourth is the spin lift, and the last term is the virtual mass force. A detailed historical development on each term in the above equation can be found in Johnson et al. (1990).

2.2 *Inclusion of the buoyancy Term*

When a single particle is immersed in a fluid which has a different density, the particle will experience a force called 'buoyancy' force. For a simple sphere immersed in a fluid medium, where the only body force is that of gravity, a simple calculation would yield

$$f_B = \rho_s V_s g + V_A \rho_f g - (V_A + V_s) \rho_f g \tag{15}$$

where f_B is the net buoyancy force, ρ_s is the density of the sphere, ρ_f is the density of the fluid, V_s is the volume of the sphere, V_A is the volume of the fluid above the sphere, and g is the acceleration due to gravity. This equation can be simplified to:

$$f_B = V_s g (\rho_s - \rho_f) = g (m_s - m_f) \tag{16}$$

where m_s and m_f are the mass of the sphere and the mass of an equal volume of fluid, respectively. Though it seems that the motion and behavior of a single particle in a fluid, for various cases, is understood [cf. Tchen (1947), Maxey and Riley (1983), Soo (1975), Ounis and Ahmadi (1989)], when we have an assembly of particles, i.e., a dense suspension, different forces such as drag, lift, etc. would have a different meaning. This holds true also for the 'buoyancy' force. In general, whether the buoyancy force is due to the density differences, or different temperatures giving rise to different densities (Bousinesq's approximation in the natural convection studies), or different pressure fields causing one phase to move with respect to the other phase, it seems that sometimes the buoyancy force acts in the direction of motion, as is the case of a single sphere immersed in a fluid, and sometimes it acts in other directions, possibly normal to the direction of motion, as is the case of Segre and Silberberg experiments, who called this effect 'particle migration'. Whether, this can be called a 'buoyancy' force of a 'lift' force, or... is an interesting issue; nevertheless, as part of the modeling effort one has to try to include the effect(s) of such force(s).

Jackson (1985) proposes that the term, $\nu \operatorname{div} \mathbf{T}_1$ should represent buoyancy effects of the fluid on the particles. This term was not added 'naturally' to the momentum equation and it does not appear to be consistent with the mixture theory formulation. There are, in general, few experiments that can be effectively used for modelling efforts. Most of the experiments are done for the sedimentation problems. Huang et al. (1971) have modeled the sedimentation of the blood, but they made very restrictive assumption that the particle velocity did not depend upon time or position. Whelan et al. (1971) also made experimental measurements of the concentration profile for the case of Erythrocyte (red cell) sedimentation in human blood. Soo (1967) also had experimental results for flow with sedimentation. Segre and Silberberg (1962 a,b) observed that spheres in laminar Poiseuille flow through a pipe (at low Re) accumulate in an annulus some distance from the tube axis. Following the initial observations, a number of investigators verify this 'tubular pinch' effect and attempt to explain the lateral (or lift) force acting on the spheres [cf. Johnson, et al., (1990)].

In this paper we propose to add the so-called buoyancy term in the mechanical interaction $\mathbf{f}_I$. That is, a term such as $-h(v)\,\mathrm{grad}\,p$ is added to the interaction, where $h(v)$ needs to be measured experimentally. We can assume as a first approximation that $h(v) = A_6 v$. With this, the modified form of the interaction term, $\mathbf{fm}_I$ takes the following form:

$$
\begin{aligned}
fm_I = {} & A_1 \,\mathrm{grad}\, v + A_2 F(v)(v_2 - v_I) + A_3 v(2\mathrm{tr}\, D_I^2)^{-1/4} D_I (v_2 - v_I) \\
& + A_4 v (W_2 - W_I)(v_2 - v_I) + A_5 a_{vm} - A_6 v\, \mathrm{grad}\, P,
\end{aligned}
$$

Notice that when $A_6 = 0$, we recover the equation proposed by Johnson et al., (1990). The dimensionless forms of the governing equations for two components flows based on the constitutive relations given by Equations (9), (10), and (14) are derived by Johnson et al., (1991 a,b). For a steady fully developed flow between two parallel plates, the velocity profiles and solids distribution can be assumed to have the form:

$$
\begin{aligned}
v_I &= U(y)\, i \\
v_2 &= U(y)\, i \\
v &= v(y)
\end{aligned}
\tag{18}
$$

The equations for conservation of mass, i.e., Equations (4) and (5) are satisfied automatically. Substituting (18) and using Equations (9) (10), and (17) into Equations (6) and (7) and following the procedure which was outlined in Johnson et al., (1991 a,b) with regard to approximating β_0-β_4 and non-dimensionalizing the equations, we obtain the following equations:

$$
- v)\, V'' \;\; v' V' - \frac{\partial P}{\partial X} \mathrm{Re} + C_2 \mathrm{Re} F(v)(U - V) = 0 \tag{19}
$$

$$
v' P - \frac{\partial P}{\partial Y} \;\; C_3 v |V'|^{-1/2} V'(U - V) - \frac{\rho_f}{Fr}[1 - v] = 0 \tag{20}
$$

$$B_3[(v+v^2)U''+(2v+1)v'U'] - 2C_2F(v)(U-V)+v\frac{\partial P}{\partial X} = 0 \tag{21}$$

$$B_0 v - (B_1+B_4)[2(1+v+v^2)v'v''+(2v+1)(v')^3] \\ -\frac{v\rho_s}{Fr} - C_3 v|V'|^{-1/2}V'(U-V)+v\frac{\partial P}{\partial Y}=0 \tag{22}$$

where

$$Re = \frac{\rho_o U_o L}{\mu_f} \qquad Fr = \frac{U_o^2}{Lg} \tag{23}$$

The first two equations are for the fluid component in the X and Y directions respectively. Equations (21) and (22) are for the solid particles, and ρ_o, U_o, L are reference quantities used in non-dimensionalizing the equations, and Re and Fr are the Reynolds and Froud numbers, respectively.

The buoyancy term, -v grad P vanishes in the limiting case as $v \to 0$. If $\varphi=0$, there is no fluid, which means there is no interaction between the solid and fluid.

Boundary Condition

As we can see from the system of Equations (19) - (22) we need to specify two boundary conditions for the fluid velocity V, two boundary conditions for the solid velocity U, two boundary conditions for the volume fraction v, and two boundary conditions for the fluid pressure, P. As with all the numerical studies of incompressible fluids, the pressure drop, which appears in the equations of motion as the gradient of the pressure, causes difficulties. In general, the pressure is eliminated by cross differentiating the momentum equation for the fluid, and thus eliminating the pressure term accordingly. This however, raises the order of the differential equations, and as a result there is now a need to provide additional boundary conditions on the velocity fields. Johnson et al., (1991 a,b) used this approach. In the present study, however, due to the inclusion of the so-called buoyancy force, depicted as h(v) grad P, the pressure term also appears in the momentum equations for the solid particles (Equations 19 and 20). Thus, we need to devise a numerical scheme for this issue. The details are given in Briggs (1995). Adherence boundary conditions are applied on both constituents at each plate:

$$U(-1)=U(1) = V(-1)=V(1)=0 \tag{24}$$

The mass flow rate of the mixture is also prescribed as a condition. For a two-component mixture, it is:

$$Q_m = \int_{-1}^{1} [(1-v)\rho_f V + v\rho_s U]dY \tag{25}$$

Sometimes, instead we can use a volumetric flow rate which is given by:

$$Q = \int_{-1}^{1} [(1-v)V + vU]\,dY$$

Equations (25) and (26) are related by $Q_m = \rho Q$ for a neutrally buoyant mixture, meaning where $\rho_f = \rho_s$.

For problems where gravity and buoyancy effects are important, the flow rate of the mixture should be the boundary condition to be specified, since it contains densities of the components. We have two boundary conditions for v: a value of v at a plate, and a prescribed average volume fraction, defined by:

$$N = \int_{-1}^{1} v\,dY$$

For non- symmetric solutions we need to specify N and a value for v(-1) as the boundary conditions. Then v(1) is determined from the solution, for the case of positive density difference, defined as ρ_f-ρ_s. For the case of negative density differences, ρ_s-ρ_f, v(-1) is the value obtained from the solution in the case of positive density difference problem as v(1). This is done to verify the accuracy of both solutions.

3. NUMERICAL METHOD AND RESULTS

The method used here is the collocation method. The details of using COLSYS code are described in the header of the program. The code is capable of solving mixed-order systems of boundary value problems in ordinary differential equations. Error tolerances on the volume fraction, solid velocity, fluid velocity, integral boundary conditions, and their derivatives are specified as 10^{-3}. The system of equations is rewritten as six equations including two integral boundary conditions for solution using COLSYS. Defining the variables by

$$Y1 = N, Y2 = Q, Y3 = P, Y4 = v, Y5 = v', Y6 = U, Y7 = U', Y8 = V, Y9 = V'$$

Equations (19)-(22), Equations (26 and (27) are expressed in terms of the notations defined in Equation (28).

The boundary conditions according to the defined notations are as follows (ρ_f-ρ_s>o):

$$Y1(-1) = 0, Y2(-1) = 0, Y4(-1) = \tau_1, Y6(-1) = 0, Y8(-1) = 0$$

$$Y1(1) = \tau_2, Y2(1) = \tau_3, Y6(1) = 0, Y8(1) = 0. \qquad (30)$$

For negative density differences,

$$Y1(-1)=0,\ Y2(-1)=0,\ Y6(-1)=0,\ Y8(-1)=0, \tag{31}$$

$$Y1(1)=\tau_2,\ Y2(1)=\tau_3,\ Y4(1)=\tau_1,\ Y6(1)=0,\ Y8(1)=0. \tag{32}$$

For each case, τ_3 is specified as 3.0, and τ_2 as 0.6. Although they give very good mathematical results, those numbers have to be verified experimentally. The reference value of P is specified at (o,o). The value of P is given as 1.0 for symmetric solutions and 0.1 for non-symmetric solutions. In order to evaluate the validity of the buoyancy term inserted into the mechanical interaction term, calculations are made and results are presented below.

Since we are using a similar scheme to that used by Johnson (1991), we will only present the cases where the buoyancy term is studied. That is, if we set $A_6=0$ in Equation (17) and in the subsequent equations, we recover the equations solved in Johnson et al., (1991 a,b) and we used this to check the accuracy of our numerical scheme.

Figure 1 shows volume fraction profiles for various values of dimensionless solid density. Physically, as the difference between the fluid density and the solid density gets bigger, granular particles tend to move towards the upper plate, so the boundary conditions should be specified in accordance with this buoyancy effect. Clearly, for a fixed value of the fluid density, the solid particles move away from the upper plate towards the lower plate, as the value of the solid density increases. Though the buoyancy effect does not directly influence the velocity profiles, it affects the velocities indirectly through its influence on the volume fraction. The influence of v on the velocity profiles is to cause a relative decrease in the fluid velocity in the areas of higher solid concentration and an increase in areas of lower solid concentration. These effects are illustrated in Figures 2 and 3.

Figure 4 shows volume fraction profiles for various values of dimensionless fluid density. At a fixed solid density value, the solid particles move away from the upper plate towards the lower plate, as fluid density decreases. Figures 5 and 6 show the buoyancy effect on the velocities through their influence on the volume fraction. Figure 7 shows volume fraction profiles for different density differences. For fluid density value smaller than the solid density, solid particles move towards the lower plate. This result is consistent with the form of the buoyancy interaction, which depends on the density differences. As buoyancy becomes more important, we expect to see decreasing solid concentration at the lower plate. Figures 8 and 9 illustrate the influence of the density differences on the velocity profiles. Note that the mass flow rate of the mixture is used in this case and is constant for each case. The lighter the fluid is, the faster it moves. The velocity of the solid particles also increase due to the coupling of equations. The buoyancy also affects the velocities through its influence on the volume fraction. These results are consistent with the form of the drag interaction, which depends upon v (becoming greater as v increases). Physically drag transfers momentum from the fluid to the solid particles. Those effects can be seen more clearly as the relative density difference gets bigger as shown in Figures 10 through 13.

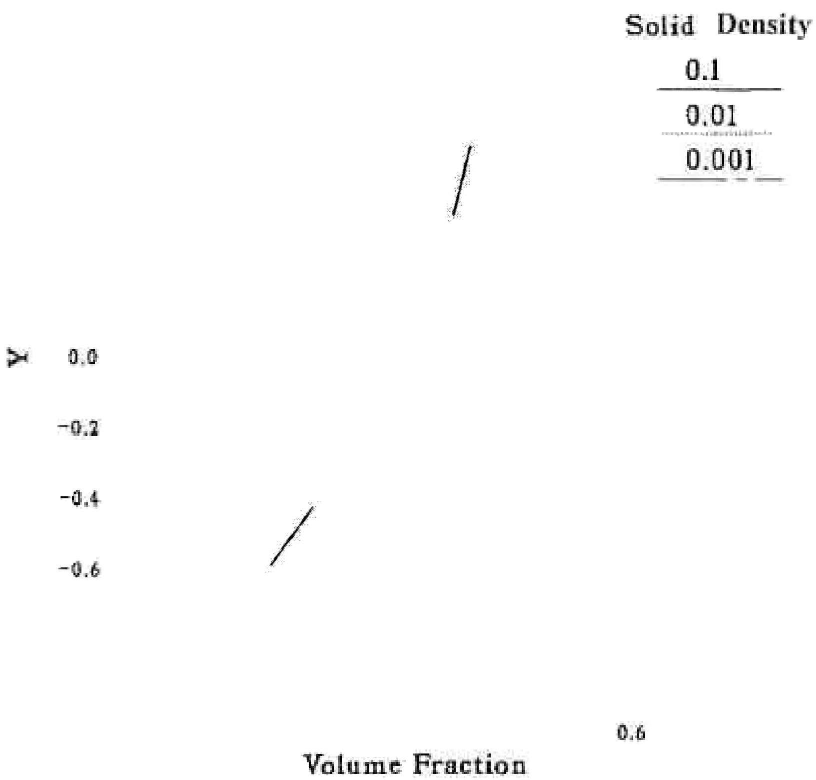

Figure = 0.1 , 0.01 ,0.001,

D_t

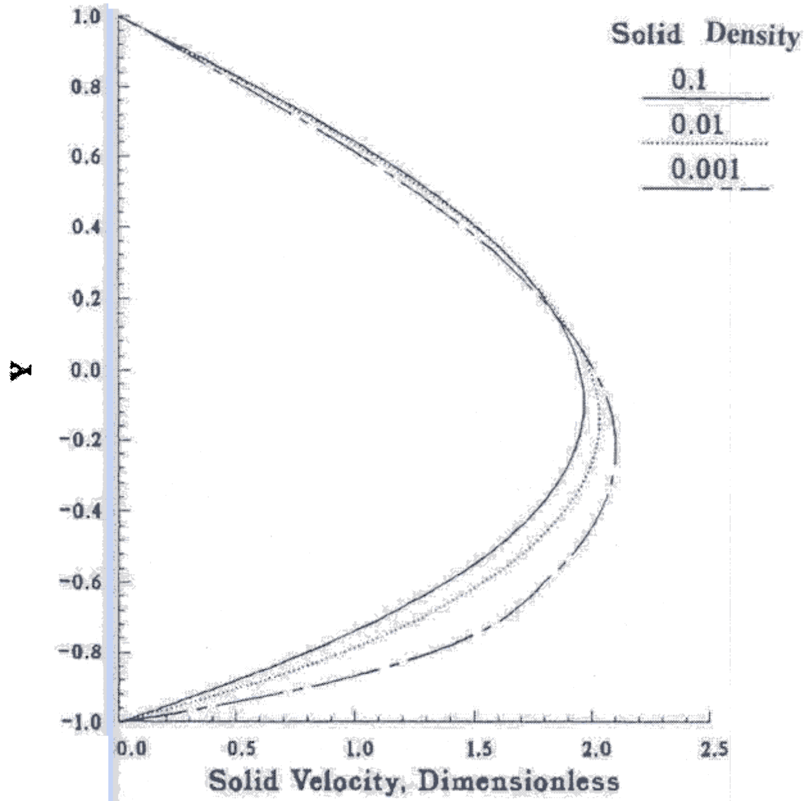

Figure 2 Effect of Bouyancy on Solid Velocity Profile; $\rho_s = 0.1$, 0.01 ,0.001, $\rho_f = 1$, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1 , A_6=1

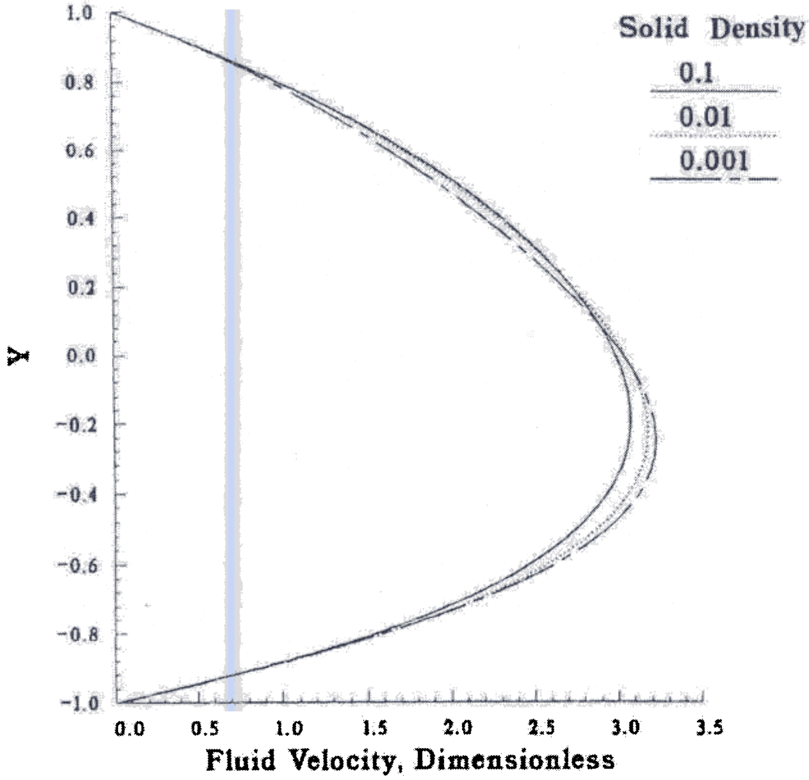

Figure 3 Effect of Bouyancy on Fluid Velocity Profile; $\rho_s = 0.1$, 0.01 ,0.001, $\rho_f = 1$, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1 ,A_6=1

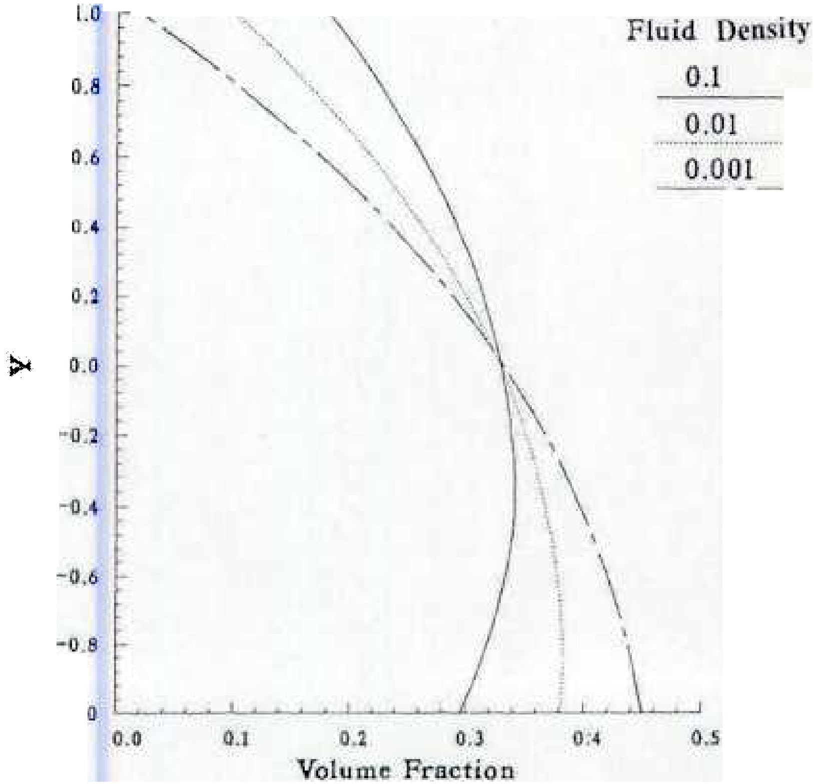

Figure 4 Effect of Bouyancy on Volume Fraction Profile; $\rho_f = 0.1$, 0.01, 0.001, $\rho_s = 1$, $C_3=0$, $L= 0$, $B_1 = -50$, $B_4 = -50$, $Re = 10$, $Fr = 1$, $B = 0.5$, $D_2 =2$ $D_1 =10$, $B_3 = 1$

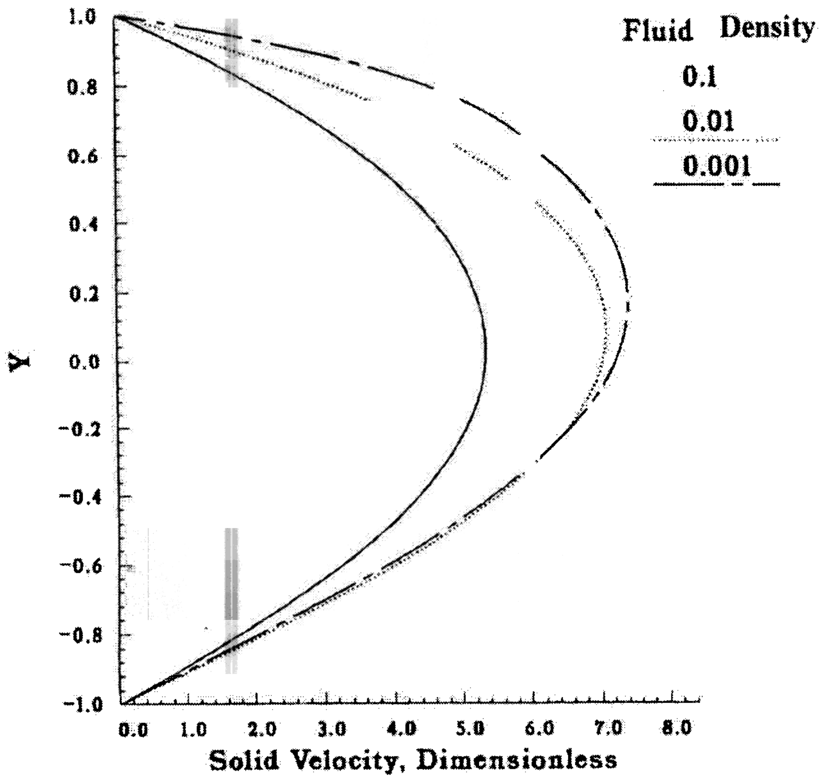

Figure 5 Effect of Bouyancy on Solid Velocity Profile; $\rho_f = 0.1$, 0.01 ,0.001, $\rho_s = 1$, $C_3=0$,$L= 0$,$B_1 = -50$, $B_4 = -50$, $Re = 10$, $Fr = 1$, $B = 0.5$, $D_2 = 2$, $D_1 = 10$, $B_3 = 1$

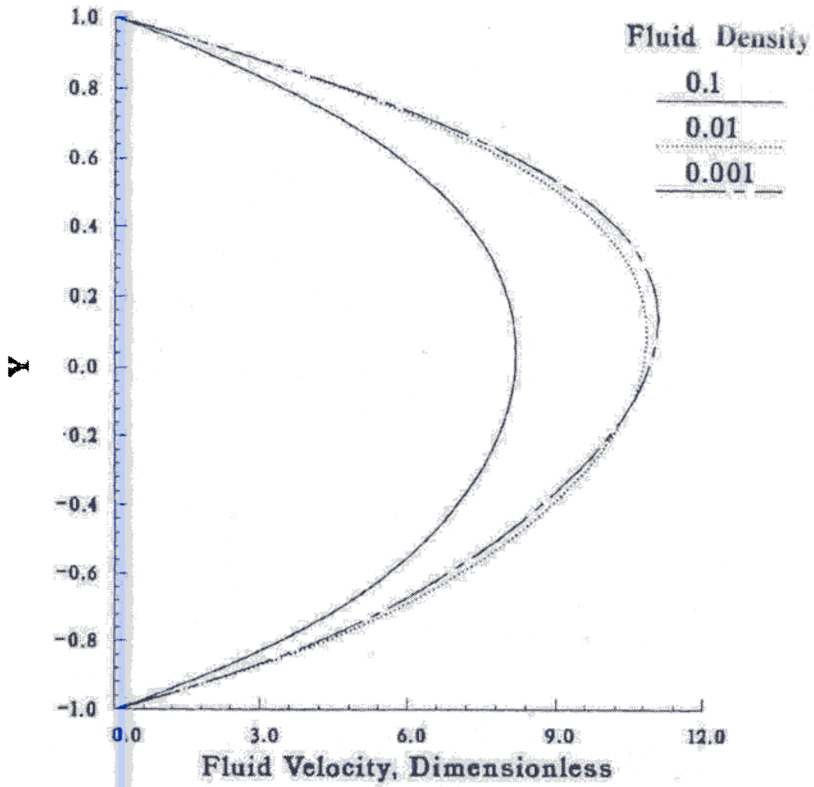

Figure 6 Effect of Bouyancy on Fluid Velocity Profile; $\rho_f = 0.1$, 0.01 ,0.001, $\rho_s = 1$, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1

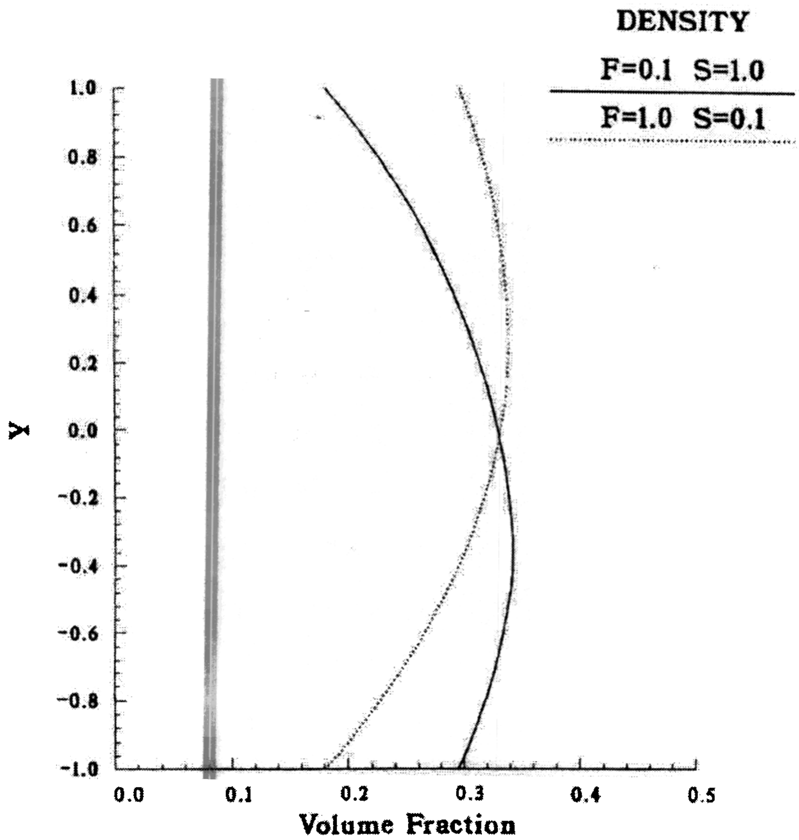

Figure 7 Effect of Bouyancy on Volume Fraction Profile; $\rho_s = 1.0, 0.1$, $\rho_f = 0.1, 1.0$, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1

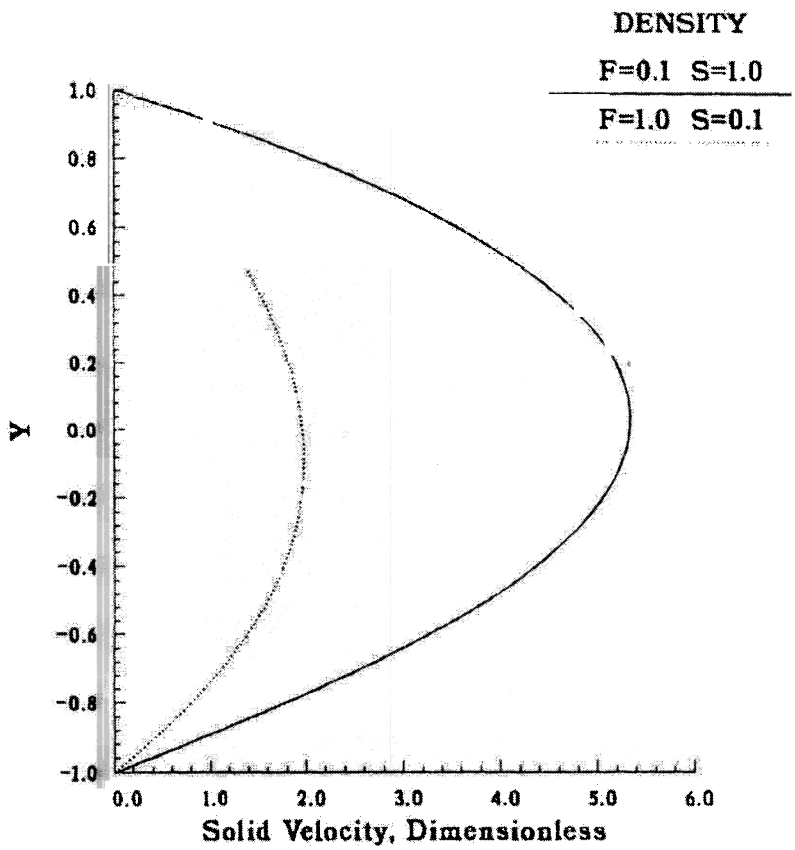

Figure 8 Effect of Bouyancy on Solid Velocity Profile; ρ_s = 1.0, 0.1 , ρ_f = 0.1, 1.0, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1

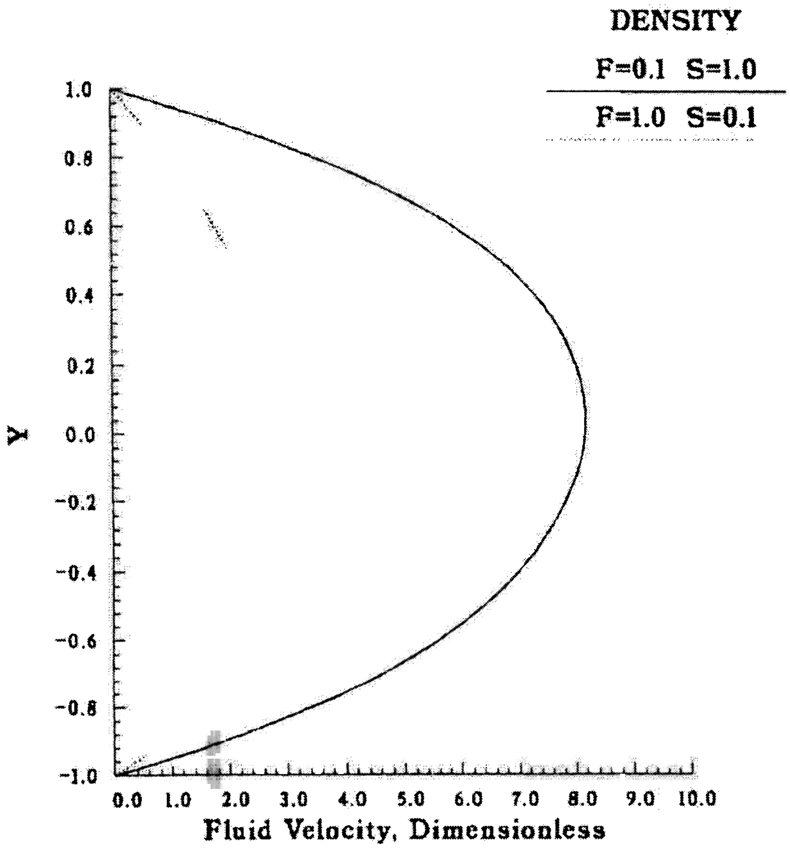

Figure 9 Effect of Bouyancy on Fluid Velocity Profile; ρ_s = 1.0, 0.1 , ρ_f = 0.1, 1.0, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_s = 1

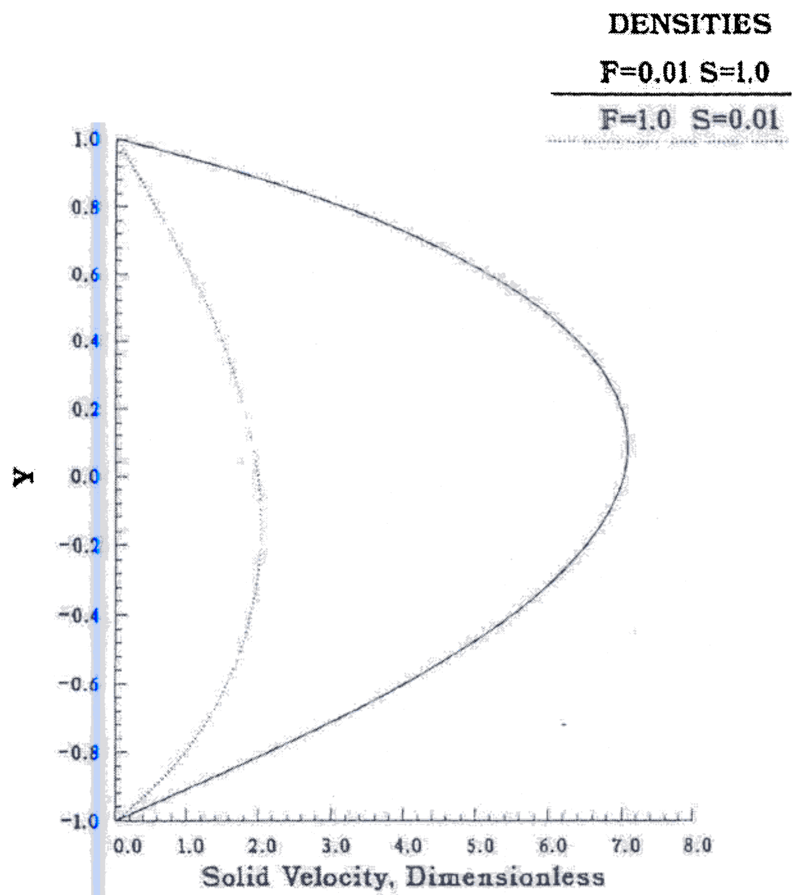

Figure 10 Effect of Bouyancy on Solid Velocity Profile; $\rho_s = 1.0,\ 0.01$, $\rho_f = 0.01,\ 1.0$, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1

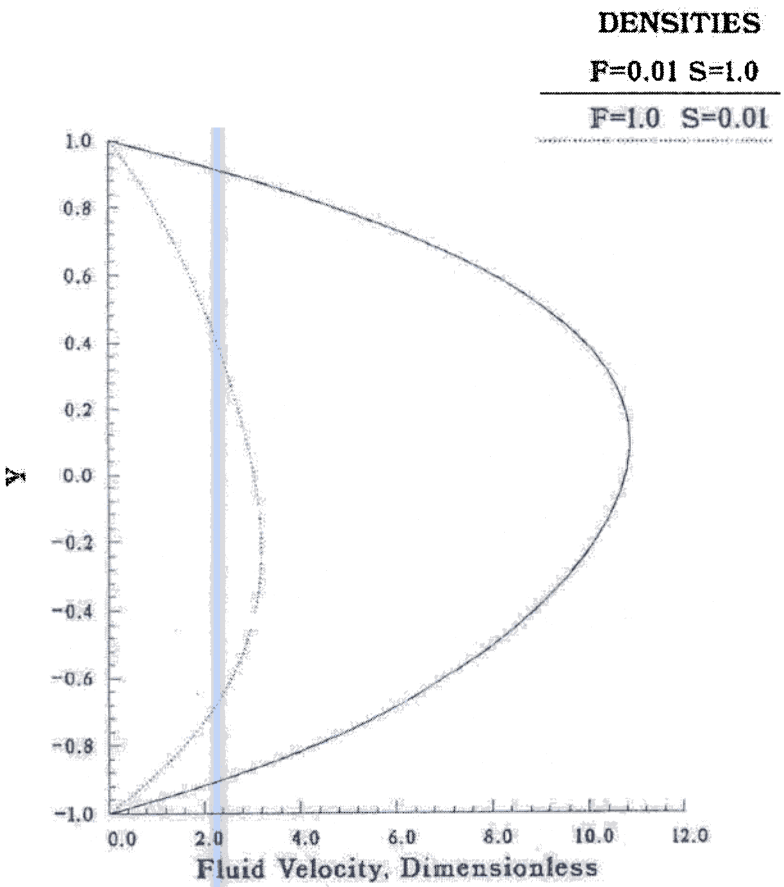

Figure 11 Effect of Bouyancy on Fluid Velocity Profile; $\rho_s = 1.0,\ 0.01$, $\rho_f = 00.1,\ 1.0$, C_3=0,L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D_1 =10, B_3 = 1

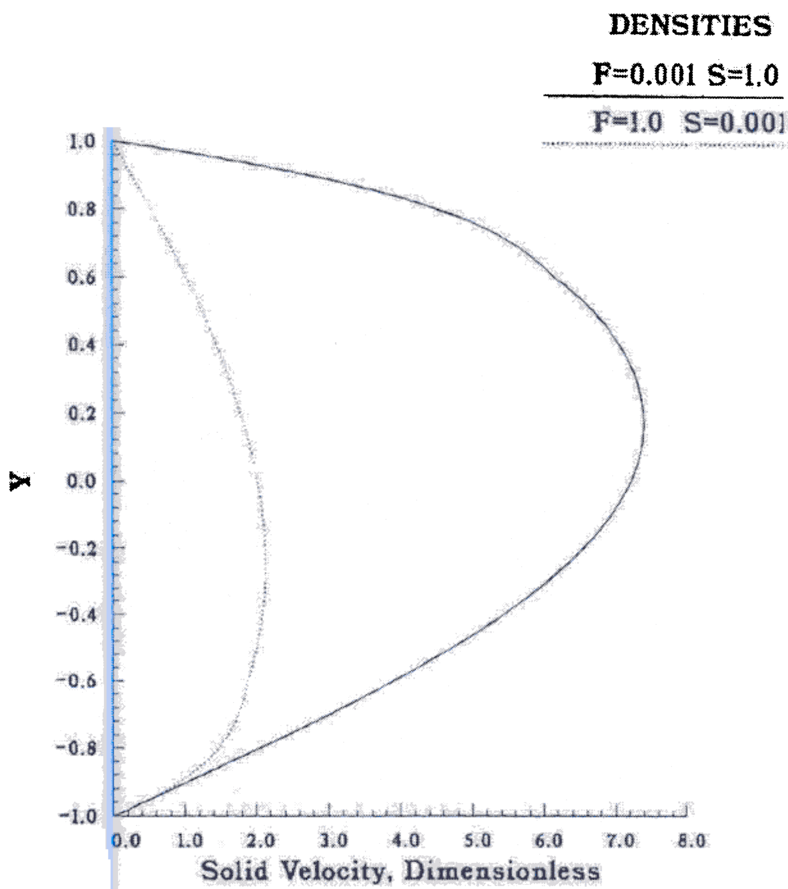

Figure 12 Effect of Bouyancy on Solid Velocity Profile; $\rho_s = 1.0,\ 0.001$, $\rho_f = 0.001$, 1.0, $C_3=0$, $L= 0$, $B_1 = -50$, $B_4 = -50$, $Re = 10$, $Fr = 1$, $B = 0.5$, $D_2 = 2$, $D_1 = 10$, $B_3 = 1$

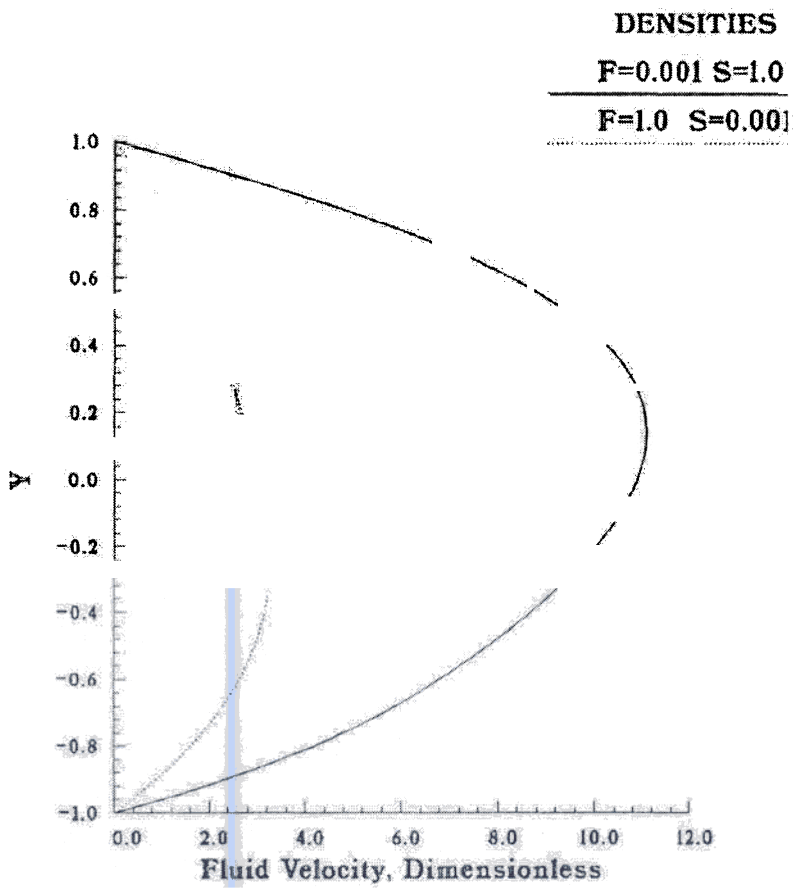

Figure 13 Effect of Bouyancy on Fluid Velocity Profile; ρ_s = 1.0, 0.001 , ρ_f = 0.001, 1.0, C_3=0, L= 0,B_1 = -50 , B_4 = -50 , Re = 10, Fr = 1, B = 0.5, D_2 =2 , D, =10, B, = 1

4. NOMENCLATURE

$\mathbf{a}$	acceleration vector
$\mathbf{a}_{vm}$	relative acceleration between components
A_i	interaction coefficients, i=1 to 5
b	body force vector
β_i	dimensionless β's, i=0 to 4
$\mathbf{B}_i$	constant part of β 's
$\mathbf{B}_i$	dimensionless parameter i=1, 2
C_i	dimensionless Ai's, i=2 to 5
C_D	dimensionless Ai's, i=2 to 5
C_H	Basset force coefficient
C_{vm}	virtual mass coefficient
$\mathbf{D}$	stretching tensor
D_i	dimensionless parameter i=1, 2
$\mathbf{f}_I$	interaction force vector
$\mathbf{F}$	deformation gradient
F	volume fraction dependence of drag
Fr	Froude number
g	gravitational acceleration
$\mathbf{I}$	identity tensor
L	characteristic length
$\mathbf{L}$	gradient of velocity tensor
L	dimensionless parameter
N	average volume fraction
p	fluid pressure
P	dimensionless fluid pressure
Q	volumetric flow rate of mixture
Q_m	mass flow rate of mixture
Re	Reynolds number
$\mathbf{T}$	stress tensor
u_o	reference velocity component
U	solid velocity
$\mathbf{v}$	velocity vector
V	fluid velocity
$\mathbf{V}$	dimensionless velocity vector
$\mathbf{W}$	spin tensor
x	direction of flow between the plates
$\mathbf{x}$	position vector
X	deformation function
$\mathbf{X}$	dimensionless position vector
Y	direction normal to plates
$\mathbf{Y}$	dimensionless Y

Greek Letters

β_i	granular solid coefficients (i=0-4)
λ_f	second coefficient of fluid viscosity
Λ	dimensionless λ_f
μ	first coefficient of fluid viscosity
ν	volume fraction of the solid
ρ	density
ρ_o	reference density
τ	dimensionless time
φ	volume fraction of fluid

Subscripts

1, f	referring to the fluid phase
2, s	referring to the solid phase
m	referring to the mixture

Superscripts

T	transpose
*	dimensionless quantity

Other Symbols

$\nabla\bullet$	divergence operator
∇	gradient operator
tr	trace of a tensor
$\otimes$	outer product
$\bullet$	dot product

5. BIBLIOGRAPHY

1) Adkins, J.E., "Non-Linear Diffusion, 1. Diffusion and Flow of a Mixture of Fluids". *Philosophical Transactions of the Royal Society of London*, 255A:607, 1963.
2) Adkins, J.E., "Non-Linear Diffusion, 2. Constitutive Equations for a Mixture of Isotropic Fluids". *Philosophical Transactions of the Royal Society of London*, 255A:635, 1963.
3) Ahmadi, G., "On Mechanics of Saturated Granular Materials". *International Journal of Non-Linear Mechanics*, 15:251, 1980.
4) Ahmadi, G., "A Continuum Theory for Two Phase Media". *Acta Mechanica*, 44:299, 1982.
5) Aoki, H., Kurosaki, Y., and Anzai, H., "Study on the Tubular Pinch Effect in Pipe Flow". *Bulletin of the Japanese Society Mechanical Engineering*, 22:206-212, 1979.
6) Ascher, U., Christianson, J., and Russel, R.D., "Collocation Software for Boundary Value ODE's". *ACM Transactions on Mathematical Software*, 7(2):209-222, June 1981.
7) Atkin, R.J. and Craine, R.E., "Continuee Theories of Mixtures" Basic Theory and Historical Development". *Quarterly Journal of Mechanics and Applied Mathematics*, 29:290, 1976.
8) Batchelor, G.K., "Sedimentation in a Dilute Dispersion of Spheres. *Journal of Fluids Mechanics*, 52:245-268, 1972.

9) Beford, A., and Drumheller, D.S., "Recent Advances: theories of Immiscible and Structured Mixtures". *International Journal of Engineering Science*, 21:863, 1983.
10) Boyle, E.J., and Massoudi, M., "A Theory for Granular Materials Exhibiting Normal Stress Effects Based on Enskong's Dense Gas Theory". *International Journal of Engineering Science*, 28: 1261, 1990.
11) Bowen, R.M., "Theory of Mixtures". *Continuum Physics*, Vol. 3: Academic Press, NY, ... 1976.
12) Briggs, A., Modifications of the Mixture theory to include Buoyancy Force. M.S. Thesis, University of Pittsburgh, 1995.
13) Drew, D.A., "Two Phase Flows:Constitutive Equations for Lift and Brownian Motion and some Basic Flows" *Archive for Rational Mechanics and Analysis*, 62:149-163, 1976.
14) Drew, D.A., "Mathematical Modeling of Two Phase Flow". *Annual Review of Fluid Mechanics*, 15:261-291, 1983.
15) Goodman, M.A., and S.C. Cowin, "A Continuum Theory for Granular Materials". *Arch. Rational Mech. Anal.*, 44:249 (1972).
16) Green, A.E., and Naghdi, P.M., "On Basic Equations for Mixtures". *Quarterly Journal of Mechanics and Applied Mathematics*, 22:427, 1969.
17) Hill, C.D., Bedford, A., and Drumheller, D.S., "An Application of Mixture Theory to Particle Sedimentation". *Transactions of the ASME Journal of Applied Mechanics*, 47:261-265, 1980.
18) Homsy, G.M., El-Kaissy, M.M., and Didwania, A., "Instability Waves and the Origin of Bubbles in Fluidized Beds", Part 2:Comparison With Theory". *International Journal of Multiphase Flow*, 6:305-318, 1980.
19) Huang, C.R., "A Mathematical Model of Sedimentation Analysis Applied to Human Whole Blood". *Biorheology*, 8:157-163, 1971.
20) Hutter, K. and K.R. Rajagopal, "On flows of granular materials", *Comtinuum Mech. Thermodyn.* 6: 81-139 (1994).
21) Jakson, R., "Hydrodynamic Stability of Fluid-Particle Systems". In *Fluidization*, Edited by J. F. Davidson, R. Clift, and D. Harrsion. Academic Press, 1985.
22) Johnson, G., Massoudi, M., and Rajagopal, K.R., "A Review of Interaction Mechanisms in Fluid-Solid Flows". *Technical Report, U.S. Dept. Of Energy Report*, DOE/PETC/TR-90/9, 1990.
23) Johnson, G., Massoudi, M., and Rajagopal, K.R., "Flow of a Fluid-Solid Mixture Between Flat Plates," *Chem. Engng. Sci.*, 46:1713, 1991.
24) Johnson, G., Massoudi, M., and Rajagopal, K.R., "Flow of a Fluid Infused with Solid Particles through a Pipe". *Int. J. Engng. Sci.*, 29:649, 1991.
25) Johnson, G., *"Application of Continuum theories to Flows of Fluid-Solid Mixtures"* PhD Thesis, University of Pittsburgh, PA 1991.
26) Katsube, N., and M.M. Carroll, "The Modified Mixture Theory for Fluid-Filled Porous Materials:Theory". *ASME J. Applied Mechanics*, 87-APM-7, 1987.
27) Katsube, N., and M.M. Carroll, "The Modified Mixture Theory for Fluid-Filled Porous Materials:Applications". *ASME J. Applied Mechanics*, 87-APM-8, 1987.
28) Massoudi, M., *"Application of Mixture Theory to Fluidized Beds"*. PhD Thesis, University of Pittsburgh, PA, 1986.
29) Maxey, M.R., and J.J. Riley, "Equation of Motion for a Small Rigid Particle in Nonuniform Flow". *Phys. Fluids*, 26:883, 1983.
30) Muller, I., "A Thermodynamic Theory of Mixtures of Fluids". *Archive for Rational Mechanics and Analysis*, 28:1-39, 1968.
31) Ounis, H., and G. Ahmadi, "Motions of Small Rigid Spheres in Simulated Random Velocity Fluid". *ASCE J. Engng. Mech.*, 115:2107, 1989.

32) Panton, R.L., Incompressible Flow. *Wiley-Interscience Publ.*, 1984.
33) Passman, S.L., Nunziato, J.W. Bailey, P.B., and Reed, K.W., "Shearing Motion of a Fluid-Saturated Granular Material". *Journal of Rheology*, 30:167-192, 1986.
34) Patankar, S.V. Numerical Heat Transfer and Fluid Flow. *Hemisphere Publ. Corp.*, Washington, DC, 1980.
35) Rajagopal, K.R., and Massoudi, M., "A Method for Measuring the Material Moduli of Granular Materials:Flow in an Orthogonal Rheometer". *Technical Report, U.S. Dept. Of Energy*, DOE/PETC/TR-90/3, 1990.
36) Rajagopal, K.R., M. Massoudi, and A.S. Wineman, "Flow of Granular Materials Between Rotating Disks," *Mech. Research Comm.*, 6:629, 1994.
37) Rajagopal, K.R., Massoudi, M., and Ekmann, J.M., "Mathematical Modeling of Fluid-solid Mixtures". *Recent Developments in Structured Continua II.Longman*, pp. 236-248, 1990.
38) Rajagopal, K.R., and L. Tao, *Mechanics of Mixtures*. World Scientific Publishing Co., Ltd., New Jersey, 1995.
39) Repetti, R.V., and Leonard, E.F., "Segre-Silberberg Annulus Formation:A Possible Explanation". *Nature*, 194:1346-1348, 1964.
40) Rizk, M.A., and S.E. Elghobashi, "The Motion of a Spherical Particle Suspended in a Turbulent Flow Near a Plane Wall". *Phys. Fluids*, 28:806, 1985.
41) Savage, S.B., "Gravity Flow of Cohensionless Granular Materials in Chutes and Channels". *J. Fluid Mech.*, 92:53, 1979.
42) Segre, G., and A. Silberberg, "Behavior of Macroscopic Rigid Spheres in Poiseuille Flow". Part 1, *J. Fluid Mech.*, 14:115, 1962.
43) Segre, G., and A. Silberberg, "Behavior of macroscopic Rigid Spheres in Poiseuille Flow". Part 2, *J. Fluid Mech.*, 14:136, 1962.
44) Soo, S.L., *Fluid Dynamics of Multiphase Systems*. Blaisdell Publishing, 1967.
45) Speckhart, F.H., and Green, W.L., "A guide to Using CSMP-The Continuous Systems Modeling Program". *Prentice Hall*, Englewood Cliffs, NJ, 1976.
46) Tchen, C.M., "Mean Value and Correlation Problems Connected with the Motion of Small Particles Suspended in a Turbulent Fluid". Martinus Nijhoff, the Hague, 1947.
47) Truesdell, C., *Rational Thermodynamics*. Springer-Verlag, 2nd edition, 1984.
48) Truesdell, C., and W. Noll. The Non-Linear Field Theories of Mechanics. Scond Edition. Springer-Verlag, Heidelberg, 1992.
49) Whelan, J.A., Huang, C.R., and Copley, A.L., "Concentration Profiles in Erthrocyte Sedimentation in Human Whole Blood". *Biorheology*, 7:205-212, 1971.